# An action approach to nodal and least energy normalized solutions for NLS

## Analytical Methods in Quantum and Classical Mechanics

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# The nonlinear Schrödinger *evolution* equation

We consider the problem

$$\begin{cases} \boldsymbol{i}\partial_t\psi = -\Delta\psi - |\psi|^{p-2}\psi, & (t,x) \in [0, T[ \times \Omega, \\ \psi(t,x) = 0, & (t,x) \in [0, T[ \times \partial\Omega, \\ \psi(0,x) = \psi_0(x), & \psi_0 :\to \mathbb{C}, x \in \Omega \end{cases} \tag{$\mathrm{NLS_{evol}}$}$$

where

- $\psi : [0, T[ \times \Omega \to \mathbb{C}$, $\Omega$ *bounded* domain in $\mathbb{R}^N$, $N \geq 1$;
- $\boldsymbol{i}^2 = -1$;
- $\partial_t\psi$ is the derivative with respect to the time variable;
- $\Delta = \sum_{1\leq i\leq N} \partial_{x_i}^2$ is the Laplacian on $\Omega$;
- $p > 2$ is a real parameter.

# Conservation laws

At least formally, the $L^2$ *norm* (the *mass*)

$$\|\psi(t,\cdot)\|_{L^2}^2 := \int_\Omega |\psi(t,x)|^2 \,\mathrm{d}x$$

and the *energy*

$$E\Big(\psi(t,\cdot)\Big) := \frac{1}{2}\int_\Omega |\nabla_x \psi(t,x)|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |\psi(t,x)|^p \,\mathrm{d}x$$

are preserved during the evolution.

# Solitary wave solutions

Opposed to blow-up: *solitary waves* of the form

$$\psi(t,x) = e^{i\lambda t} u(x)$$

where $u \in H_0^1(\Omega; \mathbb{R}) = H_0^1(\Omega)$ is a solution of

$$-\Delta u + \lambda u = |u|^{p-2} u. \tag{NLS}$$

Some vocabulary:

- $\lambda \in \mathbb{R}$ is the *frequency* of the solitary wave;
- $\|u\|_{L^2}^2 = \|\psi(t,\cdot)\|_{L^2}^2$ is its *mass*.

# Two problems

**Problem**

*Given $\lambda \in \mathbb{R}$, how to find a nonzero stationary wave of frequency $\lambda$?*

**Problem**

*Given $\mu > 0$, how to find a stationary wave of mass $\mu$?*

Vocabulary: solutions with a prescribed mass are usually called *normalized solutions*.

# Two functionals

We recall that the *energy functional* is given by

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x.$$

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Given $\lambda \in \mathbb{R}$, we also define the *action functional* by

$$\begin{aligned} J_\lambda(u) &:= E(u) + \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 \,\mathrm{d}x + \frac{\lambda}{2}\int_\Omega |u|^2 \,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p \,\mathrm{d}x. \end{aligned}$$

# Variational formulations

## Proposition

*Given $2 < p < 2^*$ and $\lambda \in \mathbb{R}$, solutions of frequency $\lambda$ correspond to critical points of $J_\lambda$ on $H_0^1(\Omega)$.*

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In the case of normalized solutions, the parameter $\lambda$ in the PDE will appear as a Lagrange multiplier associated with the constraint.

# Lower boundedness of the energy functional

## Proposition

*Let* $2 < p < 2^*$ *and* $\mu > 0$. *Then:*

- *if* $2 < p < 2 + 4/N$,

$$\inf_{\mathcal{M}_\mu} E > -\infty;$$

- *if* $2 + 4/N < p < 2^*$,

$$\inf_{\mathcal{M}_\mu} E = -\infty.$$

# A classic result and two questions

**Proposition**

*When $\mu > 0$ and $2 < p < 2 + 4/N$, then minimizers for $E$ on $\mathcal{M}_\mu$ exist, have a constant sign and are normalized solutions of $(\mathrm{NLS})$. They are called energy ground states.*

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*Given $\mu > 0$ and $2 + 4/N < p < 2^*$, do there exist normalized solutions of mass $\mu$? Is there a least energy normalized solution?*

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Answers: given by the results of the talk!

# The fixed frequency case

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However, the functional $J_\lambda$ is not bounded from below on $H_0^1(\Omega)$, since if $u \neq 0$ then

$$J_\lambda(tu) = \frac{t^2}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{\lambda t^2}{2}\|u\|^2_{L^2(\Omega)} - \frac{t^p}{p}\|u\|^p_{L^p(\Omega)} \xrightarrow[t\to+\infty]{} -\infty.$$

# The Nehari manifold

A common strategy is to introduce the *Nehari manifold* $\mathcal{N}_\lambda$, defined by

$$\begin{aligned}\mathcal{N}_\lambda &:= \left\{u \in H_0^1(\Omega) \setminus \{0\} \mid J_\lambda'(u)[u] = 0\right\} \\ &= \left\{u \in H_0^1(\Omega) \setminus \{0\} \mid \|\nabla u\|_{L^2(\Omega)}^2 + \lambda \|u\|_{L^2(\Omega)}^2 = \|u\|_{L^p(\Omega)}^p\right\}.\end{aligned}$$

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If $u \in \mathcal{N}_\lambda$, then

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{L^p(\Omega)}^p.$$

In particular, $J_\lambda$ is bounded from below on $\mathcal{N}_\lambda$.

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In particular, $J_\lambda$ is bounded from below on $\mathcal{N}_\lambda$.

## Proposition

*Given $\lambda > -\lambda_1(\Omega)$ and $2 < p < 2^*$, then minimizers for $J_\lambda$ on $\mathcal{N}_\lambda$ exist, have a constant sign and are solutions of $(\mathrm{NLS})$ having frequency $\lambda$. They are called action ground states.*

# Nodal action ground states

One defines the nodal Nehari set by

$$\mathcal{N}_\lambda^{nod} := \left\{ u \in H_0^1(\Omega) \mid u^\pm \in \mathcal{N}_\lambda(\Omega) \right\}.$$

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**Theorem (Castro, Cossio, Neuberger 1997; Bartsch-Weth 2003)**

*Given $\lambda > -\lambda_2(\Omega)$ and $2 < p < 2^*$, then minimizers for $J_\lambda$ on $\mathcal{N}_\lambda^{nod}$ exist, have two nodal zones and are solutions of $(\mathrm{NLS})$ having frequency $\lambda$. They are called nodal action ground states.*

# Comparison of the two settings so far

Abbreviation: "ground state" $\rightarrow$ GS

| | $2 < p < 2 + 4/N$ | $2 + 4/N < p < 2^*$ |
|---|---|---|
| Positive solution | Energy GS | ? |
| Sign-changing solution | ? | ? |

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| Positive solution | Action GS | Action GS |
| Sign-changing solution | Nodal action GS | Nodal action GS |

The fixed action $\lambda$ case

# Action versus energy ground states (continued)

## Theorem (Dovetta-Serra-Tilli 2022)

*Let* $2 < p < 2 + 4/N$ *and* $\Omega$ *be bounded.*

*For any* $\mu > 0$, *define*

$$\mathcal{E}(\mu) := \inf_{u \in \mathcal{M}_\mu} E(u)$$

*and, for every* $\lambda \in \mathbb{R}$, *define*

$$\mathcal{J}(\lambda) := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u).$$

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$$\mathcal{J}(\lambda) := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u).$$

*Then,* $-\mathcal{E}(2\mu)$ *is the Legendre-Fenchel transform of* $\mathcal{J}$*. Namely, one has*

$$-\mathcal{E}(2\mu) = \sup_{\lambda \in \mathbb{R}} \Big( \lambda\mu - \mathcal{J}(\lambda) \Big).$$

# Main message

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## Main message

The convex duality we just saw is a *method* !!!

More precisely:

- using such a "convex duality argument" from the action ground states when $2 + 4/N < p < 2^*$ will *also* produce normalized solutions;
- doing so from the nodal action GS will produce sign-changing normalized solutions, which is new for all $2 < p < 2^*$.

# Our result (for positive solutions)

## Theorem (De Coster-Dovetta-G.-Serra 2025)

*Let $\Omega \subset \mathbb{R}^N$ be open and bounded and, for every $2 < p < 2^*$, let*

$$M_p := \left\{ \|u\|^2_{L^2(\Omega)} \;\middle|\; u \in \mathcal{N}_\lambda(\Omega) \text{ and } J_\lambda(u) = \mathcal{J}(\lambda) \text{ for some } \lambda \in \mathbb{R} \right\}$$

*be the set of masses of all action ground states. Then,*

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*be the set of masses of all action ground states. Then,*

(i) *if $2 < p < 2 + 4/N$, then $M_p(\Omega) = (0, +\infty)$;*

(ii) *if $2 + 4/N < p < 2^*$, then there exist $0 < \mu_p < +\infty$ such that $M_p = (0, \mu_p]$.*

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This would be true *if* the map $\lambda \mapsto u_\lambda$ mapping $\lambda$ to the action GS had good continuity properties, *which is expected to be wrong in general!*

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This would be true *if* the map $\lambda \mapsto u_\lambda$ mapping $\lambda$ to the action GS had good continuity properties, *which is expected to be wrong in general!*

In fact, this map is not even well-defined as action GS might not be unique.

# A miracle

## Proposition (Key proposition)

*Let $\mu > 0$ and $2 < p < 2^*$. Assume that $\lambda_* > -\lambda_1(\Omega)$ is a local **minimum** of the map $f_\mu : [-\lambda_1, +\infty) \to \mathbb{R}$ defined by*

$$f_\mu(\lambda) := \mathcal{J}(\lambda) - \tfrac{1}{2}\mu\lambda.$$

*Then, $\mathcal{J}$ is differentiable for $\lambda = \lambda_*$ and one has that $\mathcal{J}'(\lambda_*) = \mu$, so that all action ground states with $\lambda = \lambda_*$ have mass $\mu$.*

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Our proof does not work for other types of critical points of $f_\mu$.

Here, *minimizing is better than looking for any critical points!*

*Grazie mille!*

# References

## Sign-changing normalized solutions

De Coster C., Dovetta S., Galant D., Serra E.
*An action approach to nodal and least energy normalized solutions for nonlinear Schrödinger equations.* ArXiV preprint: https://arxiv.org/abs/2411.10317 (2024).

Jeanjean L., Song L.
*Sign-changing prescribed mass solutions for $L^2$-supercritical NLS on compact metric graphs.* ArXiV preprint: https://arxiv.org/abs/2501.14642 (2025).

# References

## The nodal Nehari set

Castro A., Cossio J., Neuberger J.M,
*A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math. **27**(4), 1041–1053 (1997).

Bartsch T., Weth T.,
*A note on additional properties of sign changing solutions to superlinear elliptic equations*, Top. Meth. Nonlin. Anal. **22**, 1–14 (2003).

Szulkin A., Weth T.,
*The method of Nehari manifold*, Handbook of Nonconvex Analysis and Applications, D.Y. Gao and D. Motreanu eds., International Press, Boston, 597–632 (2010).

# References

## The $L^2$-supercritical case: unbounded domains


Jeanjean L.,
*Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlin. Anal. **28**(10), 1633–1659 (1997).

Bartsch T., de Valeriola S.
*Normalized Solutions of Nonlinear Schrödinger Equations.* Archiv der Mathematik, 100, 75–83 (2013).

Chang X., Jeanjean L., Soave N.
*Normalized solutions of $L^2$–supercritical NLS equations on compact metric graphs*, Ann. Inst. H. Poincaré (C) An. Non Lin., (2022).

Borthwick J., Chang X., Jeanjean L., Soave N.,
*Normalized solutions of $L^2$–supercritical NLS equations on noncompact metric graphs with localized nonlinearities*, Nonlinearity **36** (2023), 3776–3795.

# References

## The $L^2$-supercritical case: bounded domains

Noris B., Tavares H., Verzini G.,
*Existence and orbital stability of the ground states with prescribed mass for the $L^2$–critical and supercritical NLS on bounded domains*, Anal. PDE **7**(8), 1807–1838 (2014).

Pierotti D., Verzini G.,
*Normalized bound states for the nonlinear Schrödinger equation in bounded domains*, Calc. Var. PDE **56**, art. n. 133 (2017).

Pierotti D., Verzini G., Yu J.,
*Normalized solutions for Sobolev critical Schrödinger equations on bounded domains*, arXiv preprint 2404.04594 (2024).

# References

## Action versus energy

Jeanjean L., Lu S.-S,
*On global minimizers for a mass constrained problem*, Calc. Var. PDE **61**(6), art. n. 214 (2022).

Dovetta S., Serra E., Tilli P.,
*Action versus energy ground states in nonlinear Schrödinger equations*, Math. Ann. **385**, 1545–1576 (2023).